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Bethe ansatz at $q = 0$ and periodic box–ball systems

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Abstract
A class of periodic soliton cellular automata is introduced associated with crystals of non-exceptional quantum affine algebras. Based on the Bethe ansatz at $q = 0$, we propose explicit formulae for the dynamical period and the size of certain orbits under the time evolution in the $A_n^{(1)}$ case.

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1. Introduction

The box–ball system [TS, T] is a soliton cellular automaton on a one-dimensional lattice. It is an ultradiscrete integrable system [TTMS] that exhibits factorized scattering and has been studied from a variety of aspects. Among them, an efficient viewpoint is a solvable vertex model in statistical mechanics [B] at $q = 0$, where the time evolution of the box–ball system is identified with the action of a transfer matrix. It has led to a direct formulation [HHIKTT, FOY] by the crystal base theory, a theory of quantum group at $q = 0$ [K], and generalizations associated with quantum affine algebras [HKT1, HKOTY]. For some latest developments along this line, see [IKO, KOY]. These studies are based on the idea of commuting transfer matrices [B]. As a method of analysing solvable lattice models, it is complementary to the most efficient technique known as the Bethe ansatz [Be]; therefore, it is natural to seek its application to the box–ball system and its generalizations.

The aim of this paper is to extend the box–ball system to periodic versions and launch a Bethe ansatz approach to them. For non-exceptional affine Lie algebra $\mathfrak{g}_n$, we construct a periodic ultradiscrete dynamical system that tends to the $\mathfrak{g}_n$ automaton [HKT1] in an infinite lattice limit. Here is an example of the time evolution pattern for $\mathfrak{g}_n = A_2^{(1)}$:

$$\begin{aligned} t = 0: &\quad 1\,1\,2\,1\,3\,2 \\ t = 1: &\quad 3\,2\,1\,2\,1\,1 \\ t = 2: &\quad 1\,1\,3\,1\,2\,2 \\ t = 3: &\quad 2\,2\,1\,3\,1\,1 \\ t = 4: &\quad 1\,1\,2\,2\,3\,1 \end{aligned}$$

$$
\begin{aligned}
t = 5: &\quad 2\,1\,1\,1\,2\,3\\
t = 6: &\quad 1\,3\,2\,1\,1\,2\\
t = 7: &\quad 2\,1\,1\,3\,2\,1\\
t = 8: &\quad 1\,2\,2\,1\,1\,3\\
t = 9: &\quad 3\,1\,1\,2\,2\,1\\
t = 10: &\quad 2\,3\,1\,1\,1\,2\\
t = 11: &\quad 1\,2\,3\,2\,1\,1\\
t = 12: &\quad 1\,1\,2\,1\,3\,2.
\end{aligned}
$$

Regarding the letter 1 as background, one observes two solitons proceeding cyclically to the right with velocity = amplitude equal to 2 and 1. They repeat collisions (or overtaking) under which the reactions $32 \times 2 \to 3 \times 22$ and $22 \times 3 \to 2 \times 32$ take place. Behind such dynamics there underlies a solvable vertex model at $q = 0$, where only some selected configurations have non-zero Boltzmann weights and the transfer matrix yields a deterministic evolution of the spins on one row to another. For instance, the transition from $t = 0$ to $t = 2$ states has been determined from the configuration in figure 1 on a two-dimensional square lattice.

This is a configuration of the fusion $U_q\big(A_2^{(1)}\big)$ vertex model that survives at $q = 0$. In the terminology of quantum inverse scattering method [STF], the quantum space on vertical lines carries the fundamental representation (1, 2 or 3) of $A_2 = sl_3$ and the auxiliary space on horizontal lines does the three-fold symmetric tensor representation $(111, 112, \ldots, 333)$. The automaton states live on the vertical lines. The dynamics is governed by *combinatorial* R, which is the quantum R matrix at $q = 0$ specified by local configurations around a vertex. The states on horizontal lines are so chosen that they become equal at the both ends reflecting the periodic boundary condition.

In this paper, we introduce analogous periodic automata for any non-exceptional affine Lie algebra $\mathfrak{g}_n$ based on the factorization of the combinatorial R [HKT2]. They may be viewed as the system of particles that undergo pair creation and annihilation though the collisions. Moreover, we exploit how the Bethe ansatz at $q = 0$ [KN] yields the dynamical period and size of certain orbits. For instance, in the above time evolution pattern, $t = 0$ and $t = 12$ states are identical, hence the dynamical period is 12. We propose the general formula (16) for the dynamical period in the $A_n^{(1)}$ case, which indeed predicts 12 in the above example. It is expressed as a least common multiple of the rational numbers arising from Bethe eigenvalues at $q = 0$.

In [KN], the Bethe equation is linearized into the string centre equation and an explicit character formula (23) has been established by counting off-diagonal solutions to the string centre equation. It is a version of fermionic formulae and is called the combinatorial completeness of the Bethe ansatz at $q = 0$. In (24), we relate each summand (22) in the character formula to the size of a certain orbit under the time evolution. Such a result will be useful to study the entropy of the automata.

The formulae for the dynamical period (16) and the orbit size (24) are novel applications of the Bethe ansatz to ultradiscrete integrable systems. Upon identification of strings in the Bethe ansatz with solitons in the automata, they reproduce the expressions in [YYT] for $A_1^{(1)}$ with $l = \infty$. In our approach, we also use the combinatorial Bethe ansatz at $q = 1$ [KKR, KR], namely rigged configurations and their bijective correspondence with automaton highest states. In this terminology, it is the configuration that plays the role of the conserved quantity, which is an analogous feature to the infinite system [KOTY]. It is an interesting problem to synthesize the combinatorial Bethe ansätze at $q = 1$ and $q = 0$, which will provide a unified perspective on the automata on the infinite and the periodic lattices.

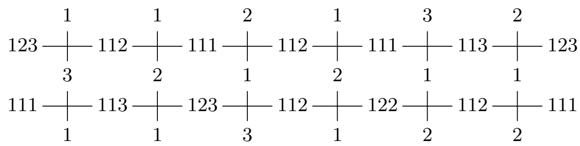

Figure 1. A vertex configuration.

The paper is arranged as follows. In section 2, a periodic $\mathfrak{g}_n$ automaton is introduced. It tends to that in [HKT1] in an infinite lattice limit and includes that in [YT, MIT] as a $\mathfrak{g}_n = A_n^{(1)}$ case. In section 3, the Bethe eigenvalues are investigated at $q = 0$. In section 4, the dynamical period of the periodic automata is related to the Bethe eigenvalue studied in section 3. In section 5, sizes of certain orbits are related to the character formula in [KN]. In the last two sections, conjectures are presented with compelling experimental data. The states treated there are time evolutions of the highest ones. The classes of time evolutions being considered in sections 4 and 5 are different. In fact, the latter is wider containing the former; therefore, the 'period' in section 4 is a notion different from the 'size of orbit' in section 5. A more unified framework including a treatment of non-highest states will be presented elsewhere. The last table in section 4 is a preliminary report on $D_4^{(1)}$. For standard notations and facts in the crystal base theory, we refer to [K, KKM, KMN].

2. Periodic $\mathfrak{g}_n$ automaton

Let $U_q(\mathfrak{g}_n)$ be the quantum affine algebra associated with non-exceptional $\mathfrak{g}_n = A_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ and $D_{n+1}^{(2)}$. Denote by B_l the crystal of the l-fold symmetric fusion of the vector representation of $U_q(\mathfrak{g}_n)$ [KKM]. We are going to introduce a dynamical system on the finite tensor product $B := B_{l_1} \otimes B_{l_2} \otimes \cdots \otimes B_{l_L}$. An element of B will be called a *path*. The representative time evolution is given by

$$T_\infty = \sigma_B S_{i_d} \cdots S_{i_2} S_{i_1}. \tag{1}$$

Here S_i is the Weyl group operator [K] and $\sigma_B = \overbrace{\sigma \otimes \cdots \otimes \sigma}^{L}$ with each σ acting on the components B_{l_i} individually according to table 1. For example, for the element $11245 \in B_5$ of $A_4^{(1)}$ represented by the semi-standard tableau, one has $\sigma(11245) = 13455$. See [HKT2], section 2 for the notation in the other algebras. We call the dynamical system on B with the time evolution (1) the periodic $\mathfrak{g}_n$ automaton. In case B is of the form $B = B_1^{\otimes L}$, it will be called the basic periodic $\mathfrak{g}_n$ automaton.

Let us illustrate (1) along a $\mathfrak{g}_n = A_2^{(1)}$ example. The time evolution $T_\infty = \sigma_B S_2 S_0$ of the $t = 0$ path $112132 \in B_1^{\otimes 6}$ into 3 2 1 2 1 1 at $t = 1$ in section 1 is computed as

$$\begin{array}{rcccccccc} & 1\,1\,2\,1\,3\,2 & \overset{S_0}{\mapsto} & 1\,3\,2\,3\,3\,2 & \overset{S_2}{\mapsto} & 1\,3\,2\,3\,2\,2 & \overset{\sigma_B}{\mapsto} & 3\,2\,1\,2\,1\,1. \\ \text{0-signature} & -\,-\;\;\;-\,+ & & -\,+\;\;\;+\,+ & & -\,+\;\;\;+ & & \\ \text{2-signaure} & (+\;\;\;-)+ & & -(+-)-+ & & -(+-)++ & & \end{array} \tag{2}$$

For the first three paths, we have exhibited the 0-signature and 2-signature. In general, the i-signature of an element a in the $A_2^{(1)}$ crystal $B_1 = \{1, 2, 3\}$ is the symbol $+$ if $a = i$, $-$ if $a = i+1$ mod 3 and none otherwise. From the array of i-signatures, one eliminates the pair $+-$ (not $-+$) successively to finally reach the pattern $\overbrace{-\cdots-}^{\alpha}\overbrace{+\cdots+}^{\beta}$ called the reduced i-signature. Then the action of S_i is unambiguously defined as the interchange $\overbrace{-\cdots-}^{\alpha}\overbrace{+\cdots+}^{\beta} \mapsto \overbrace{-\cdots-}^{\beta}\overbrace{+\cdots+}^{\alpha}$ on

Table 1. The data σ, d and i_k.

$\mathfrak{g}_n$	σ	d	$i_d, \ldots, i_1$	W
$A_n^{(1)}$	$a \mapsto a-1$	n	$2, 3, \ldots, n-1, n, 0$	$W(A_{n-1})$
$A_{2n-1}^{(2)}$	$1 \leftrightarrow \bar{1}$	$2n-1$	$0, 2, \ldots, n-1, n, n-1, \ldots, 2, 0$	$W(BC_{n-1})$
$A_{2n}^{(2)}$	id	$2n$	$1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1, 0$	$W(BC_{n-1})$
$B_n^{(1)}$	$1 \leftrightarrow \bar{1}$	$2n-1$	$0, 2, \ldots, n-1, n, n-1, \ldots, 2, 0$	$W(BC_{n-1})$
$C_n^{(1)}$	id	$2n$	$1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1, 0$	$W(BC_{n-1})$
$D_n^{(1)}$	$1 \leftrightarrow \bar{1}, n \leftrightarrow \bar{n}$	$2n-2$	$0, 2, \ldots, n-2, \left\{ {n-1, n \atop n, n-1} \right\}, n-2, \ldots, 2, 0$	$W(D_{n-1})$
$D_{n+1}^{(2)}$	id	$2n$	$1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1, 0$	$W(BC_{n-1})$

the reduced i-signature. In (2), we have shown the elimination of the $+-$ pairs by parentheses. By the very same rule, the Weyl group operators in general $\mathfrak{g}_n$ and $B = B_{l_1} \otimes \cdots \otimes B_{l_L}$ can be computed using the necessary data on B_l in [KKM].

One may question the relation between the two derivations of the time evolution $112132(t = 0) \mapsto 321211(t = 1)$, one as (2) and the other as in figure 1. Let us clarify it by explaining the origin of (1). Recall that the automata in the infinite system [HKT1, HHIKTT, FOY, HKOTY] have the set of states $\cdots \otimes B_{l_i} \otimes B_{l_{i+1}} \otimes \cdots$ with the boundary condition that the sufficiently distant local states are the highest element $u_{l_i} = (1^{l_i}) \in B_{l_i}$. The commuting family of time evolutions $T_l (l \in \mathbb{Z}_{\geqslant 1})$ is induced by the relation

$$\begin{gathered} B_l \otimes (\cdots \otimes B_{l_i} \otimes B_{l_{i+1}} \otimes \cdots) \simeq (\cdots \otimes B_{l_i} \otimes B_{l_{i+1}} \otimes \cdots) \otimes B_l \\ u_l \otimes p \simeq T_l(p) \otimes u_l \end{gathered} \tag{3}$$

under the isomorphism of crystals. It was proved in [HKT2] that T_l with sufficiently large l is factorized as (1), where all S_i actually act as $\tilde{e}_i^{\infty}$. In this sense, (1) is a natural analogue of T_∞ in the infinite system, which corresponds to the limit of the periodic $\mathfrak{g}_n$ automaton when the system size L grows to infinity under the above-mentioned boundary condition. The product (1) is a translation in the extended affine Weyl group. The indices i_k in table 1 are equal to i_{k+j} in [HKT2] for some j. They have been chosen so that the tableau letter representing the background or 'empty box' becomes 1.

Let us comment on the analogue of the time evolution T_l with finite l on our periodic $\mathfrak{g}_n$ automaton. A natural idea is to define it by an analogue of relation (3) as $v_l \otimes p \simeq p' \otimes v_l$, where $v_l \in B_l$ is not necessarily the highest element u_l in general. If such a v_l exists and p' is unique even when v_l is not unique, we set $T_l(p) = p'$ and say that $T_l(p)$ exists. $T_l(p)$ does not always exist. For instance, in the $A_n^{(1)}$ case, v_1 does not exist for $L = l = 1$, $p = 12 \in B = B_2$, and p' is not unique for $L = 2, l = 1, p = 12 \otimes 12 \in B_2 \otimes B_2$. See section 5 for more arguments. On the other hand, for l sufficiently large we expect that $T_l(p)$ exists. In fact, the following assertion is valid.

Theorem 1. *Let $\mathfrak{g}_n = A_n^{(1)}$ (hence $d = n$). Pick any element $p \in B$ such that*

$$\varphi_{i_k}(S_{i_{k-1}} \cdots S_{i_1}(p)) \leqslant \varepsilon_{i_k}(S_{i_{k-1}} \cdots S_{i_1}(p)) \qquad \text{for } 1 \leqslant k \leqslant d. \tag{4}$$

Set $v_l = (x_1, \ldots, x_{n+1})$. Here the number $x_i \in \mathbb{Z}_{\geqslant 0} (i \in \mathbb{Z}_{n+1})$ of the letter i in the semi-standard tableau on length l row is determined by $x_{i_k} = \varphi_{i_k}(S_{i_{k-1}} \cdots S_{i_1}(p))$ for $1 \leqslant k \leqslant n$ and $x_1 + \cdots + x_{n+1} = l$, which is possible for l large. Then for sufficiently large l, the relation

$$v_l \otimes p \simeq T_\infty(p) \otimes v_l \tag{5}$$

holds under the isomorphism of crystals $B_l \otimes B \simeq B \otimes B_l$ with T_∞ given by (1).

Condition (4) stated in an intrinsic manner is actually a simple postulate that among $\{1, \ldots, n+1\}$ the letter 1 should be no less than any other ones in the semi-standard tableaux consisting of p. On the paths in figure 1 ($112132, 321211, 113122 \in B_1^{\otimes 6}$), one has $T_3 = T_\infty$. A similar theorem is valid also for $D_n^{(1)}$.

The time evolution T_∞ (1) commutes with several operators acting on B, which form the symmetry of (T_∞ flow of) our periodic $\mathfrak{g}_n$ automaton. By using (1) and $S_i\sigma_B = \sigma_B S_{\sigma^{-1}(i)}$ (see [HKT2]), it is easy to check that $T_\infty S_i = S_i T_\infty$ for $i \neq 0, 1$. The symmetry operators or 'Bäcklund transformations' $\{S_i \mid 2 \leqslant i \leqslant n\}$ form a classical Weyl group listed in the rightmost column of table 1. This is a smaller symmetry compared with the $U_q(\overline{\mathfrak{g}}_{n-1})$-invariance in the case of the infinite system [HKOTY].

Here is an example of the time evolution (downward) in the periodic $A_3^{(1)}$ automaton with $B = B_3 \otimes B_1 \otimes B_1 \otimes B_1 \otimes B_1 \otimes B_2 \otimes B_1$. At each time step, the paths connected by the Weyl group actions S_2 and S_3 are shown, forming commutative diagrams.

$$\begin{array}{lllll}
133\cdot4\cdot1\cdot3\cdot4\cdot12\cdot4 & \overset{S_2}{\mapsto} & 123\cdot4\cdot1\cdot2\cdot4\cdot12\cdot4 & \overset{S_3}{\mapsto} & 123\cdot4\cdot1\cdot2\cdot3\cdot12\cdot3 \\
124\cdot3\cdot4\cdot1\cdot3\cdot14\cdot3 & & 124\cdot3\cdot4\cdot1\cdot2\cdot14\cdot2 & & 123\cdot3\cdot4\cdot1\cdot2\cdot13\cdot2 \\
134\cdot2\cdot3\cdot4\cdot1\cdot34\cdot1 & & 124\cdot2\cdot3\cdot4\cdot1\cdot24\cdot1 & & 123\cdot2\cdot3\cdot4\cdot1\cdot23\cdot1 \\
134\cdot1\cdot2\cdot2\cdot4\cdot13\cdot4 & & 124\cdot1\cdot2\cdot3\cdot4\cdot12\cdot4 & & 123\cdot1\cdot2\cdot3\cdot4\cdot12\cdot3
\end{array}$$

A similar example from the basic periodic $D_4^{(1)}$ automaton with $B = B_1^{\otimes 12}$ is

$$\begin{array}{lllll}
\bar{2}2221111 4\bar{2}11 & \overset{S_2}{\mapsto} & \bar{2}3221111 4\bar{2}11 & \overset{S_4}{\mapsto} & \bar{2}\bar{4}221111\bar{3}\bar{2}11 \\
1111\bar{2}22214\bar{2}1 & & 1111\bar{2}32214\bar{2}1 & & 1111\bar{2}\bar{4}221\bar{3}\bar{2}1 \\
21111111\bar{2}24\bar{1} & & 21111111\bar{2}34\bar{1} & & 21111111\bar{2}\bar{4}\bar{3}\bar{1} \\
2\bar{2}\bar{3}421111131 & & 2\bar{2}\bar{2}421111131 & & 2\bar{2}\bar{2}\bar{3}211111\bar{4}1 \\
12111\bar{2}\bar{3}42113 & & 12111\bar{2}\bar{2}42113 & & 12111\bar{2}\bar{2}\bar{3}211\bar{4} \\
413211111\bar{2}\bar{3}2 & & 413211111\bar{2}\bar{2}2 & & \bar{3}1\bar{4}211111\bar{2}\bar{2}2 \\
3\bar{3}11\bar{3}4321111 & & 3\bar{2}11\bar{3}4321111 & & 3\bar{2}11\bar{3}\bar{3}\bar{4}21111 \\
13\bar{3}11111\bar{3}432 & & 13\bar{2}11111\bar{3}432 & & 13\bar{2}11111\bar{3}\bar{3}\bar{4}2
\end{array}$$

Let us remark on another family of maps on B, which may also be regarded as time evolutions. For $A_n^{(1)}$, it is a dual of (1) (cf [KNY]). Consider the maps $\mathcal{T}_1, \ldots, \mathcal{T}_L : B \to B$ defined by

$$\begin{aligned}
&\mathcal{T}_i = R_{i-1i} \cdots R_{23} R_{12} P_i R_{L-1L} \cdots R_{i+1i+2} R_{ii+1}, \\
&P_i : B^{\vee i} \otimes B_{l_i} \to B_{l_i} \otimes B^{\vee i} \\
&\qquad p \otimes b \mapsto b \otimes p.
\end{aligned} \tag{6}$$

Here R_{kk+1} is the combinatorial R that exchanges the kth and $(k+1)$th components from the left and $B^{\vee i} = B_{l_1} \otimes \cdot \cdot \overset{i}{\vee} \cdot \cdot \otimes B_{l_L}$ is B without the component B_{l_i}. It is an observation going back to [Y] that the Yang–Baxter equation and the inversion relation of R lead to the commuting family $\mathcal{T}_i \mathcal{T}_j = \mathcal{T}_j \mathcal{T}_i$. Note that $\mathcal{T}_i = \mathcal{T}_{i+1}$ when $l_i = l_{i+1}$.

3. Bethe eigenvalues at $q = 0$

In this section, we exclusively consider the simply laced $\mathfrak{g}_n$. Eigenvalues of row transfer matrices in trigonometric vertex models are given by the analytic Bethe ansatz [R, KS]. In the present case, the relevant quantity is the top term of $\Lambda_l^{(1)}(u)$ ((2.12) in [KS] modified with a parameter $\hbar$ to fit the notation here):

$$\frac{Q_1(u - l\hbar)}{Q_1(u + l\hbar)} \tag{7}$$

at the shift (or Hamiltonian) point $u = 0$. Here $Q_1(u) = \prod_k \sinh \pi\big(u - \sqrt{-1}u_k^{(1)}\big)$, where $\big\{u_j^{(a)}\big\}$ are to satisfy the Bethe equation (equation (2.1) in [KN]). For the string solution ([KN], definition 2.3), (7) with $u = 0$ tends to

$$\Lambda_l := \prod_{j\alpha} \big(- z_{j\alpha}^{(1)}\big)^{\min(j,l)} \tag{8}$$

in the limit $q = \exp(-2\pi\hbar) \to 0$. Here $z_{j\alpha}^{(a)}$ is the centre of the αth string having colour a and length j. Denote by $m_j^{(a)}$ the number of such strings. The product in (8) is taken over $j \in \mathbb{Z}_{\geqslant 1}$ and $1 \leqslant \alpha \leqslant m_j^{(1)}$. At $q = 0$, the Bethe equation becomes the string centre equation ([KN], equation (2.36)):

$$\prod_{(b,k)\in H} \prod_{\beta=1}^{m_k^{(b)}} \big(z_{k\beta}^{(b)}\big)^{A_{aj\alpha,bk\beta}} = (-1)^{p_j^{(a)}+m_j^{(a)}+1}, \tag{9}$$

where $H := \big\{(a, j) \mid 1 \leqslant a \leqslant n,\ j \in \mathbb{Z}_{\geqslant 1}, m_j^{(a)} > 0\big\}$ (denoted by H' in [KN]). $A_{aj\alpha,bk\beta}$ and $p_j^{(a)}$ are defined by

$$A_{aj\alpha,bk\beta} = \delta_{ab}\delta_{jk}\delta_{\alpha\beta}\big(p_j^{(a)} + m_j^{(a)}\big) + C_{ab}\min(j, k) - \delta_{ab}\delta_{jk}, \tag{10}$$

$$p_j^{(a)} = \sum_{k\geqslant 1} \min(j, k)\nu_k^{(a)} - \sum_{(b,k)\in H} C_{ab}\min(j, k)m_k^{(b)}, \tag{11}$$

where $(C_{ab})_{1\leqslant a,b\leqslant n}$ is the Cartan matrix of the classical part of $\mathfrak{g}_n$. The integer $\nu_k^{(a)}$ is the number of the Kirillov–Reshetikhin modules $W_k^{(a)}$ contained in the quantum space on which the transfer matrices act. In our case, the crystal of the quantum space is taken as $B = B_{l_1} \otimes \cdots \otimes B_{l_L}$ in section 2, hence $\nu_k^{(a)} = \delta_{a1}(\delta_{kl_1} + \cdots + \delta_{kl_L})$. To avoid a notational complexity, we temporally abbreviate the triple indices $aj\alpha$ to j, $bk\beta$ to k and accordingly $z_{k\beta}^{(b)}$ to z_k, etc. Then (8) reads

$$\Lambda_l = \prod_k (-z_k)^{\rho_k}, \tag{12}$$

where ρ_k is actually dependent on l and is given by $\rho_k = \delta_{b1}\min(k, l)$ for k corresponding to $bk\beta$. The string centre equation (9) is written as

$$\prod_k (-z_k)^{A_{j,k}} = (-1)^{s_j} \tag{13}$$

for some integer s_j. Note that $A_{j,k} = A_{k,j}$. Suppose that the $q = 0$ eigenvalue (12) satisfies $\Lambda_l^{\mathcal{P}_l} = \pm 1$ for generic solutions to the string centre equation (13). It means that there exist integers r_j such that $\sum_j r_j A_{j,k} = \mathcal{P}_l \rho_k$, or equivalently

$$r_j = \mathcal{P}_l \frac{\det A[j]}{\det A}, \tag{14}$$

where $A[j]$ denotes the matrix $A = (A_{j,k})$ with its jth column replaced by ${}^t(\rho_1, \rho_2, \ldots)$. In view of the condition $\forall r_j \in \mathbb{Z}$, the minimum integer value allowed for $\mathcal{P}_l$ is

$$\mathcal{P}_l = \mathrm{LCM}\left(1, \bigcup_k{}' \frac{\det A}{\det A[k]}\right), \tag{15}$$

where LCM stands for the least common multiple and $\cup_k'$ means the union over those k such that $A[k] \neq 0$. The determinants here can be simplified by elementary transformations (cf [KN], equation (3.9)). The result is expressed in terms of determinants of matrices with indices in H:

$$\mathcal{P}_l = \mathrm{LCM}\left(1, \bigcup_{(b,k)\in H}{}' \frac{\det F}{\det F[b,k]}\right), \tag{16}$$

where the matrix $F = (F_{aj,bk})_{(a,j),(b,k)\in H}$ is defined by

$$F_{aj,bk} = \delta_{ab}\delta_{jk} p_j^{(a)} + C_{ab} \min(j,k) m_k^{(b)}. \tag{17}$$

The matrix $F[b,k]$ is obtained from F by replacing its (b,k)th column as

$$F[b,k]_{aj,cm} = \begin{cases} F_{aj,cm}, & (c,m) \neq (b,k), \\ \delta_{a1} \min(j,l), & (c,m) = (b,k). \end{cases} \tag{18}$$

The union in (16) is taken over those (b,k) such that $\det F[b,k] \neq 0$. See also the remark before conjecture 1 in section 4.

The LCM in (16) can further be simplified when $\mathfrak{g}_n = A_1^{(1)}$ and $\nu_j^{(1)} = L\delta_{j1}$. We write $m_j^{(1)}, p_j^{(1)}, F[1,k]$ just as $m_j, p_j, F[k]$ and parameterize the set $H = \{j \in \mathbb{Z}_{\geqslant 1} \mid m_j > 0\}$ as $H = \{(0 <)J_1 < \cdots < J_s\}$. The matrix $F[k]$ is obtained by replacing the kth column of F by ${}^t(\min(J_1,l), \min(J_2,l), \ldots)$. A direct calculation leads to

$$\det F = p_{J_0} p_{J_1} \cdots p_{J_{s-1}}, \tag{19}$$

$$\det F[k+1] - \det F[k] = \frac{p_{J_0} p_{J_1} \cdots p_{J_{s-1}} p_{i_s} (i_{k+1} - i_k)}{p_{i_{k+1}} p_{i_k}}, \qquad 0 \leqslant k \leqslant s-1, \tag{20}$$

where we have set $i_k = \min(J_k, l)$, $J_0 = 0$, $i_0 = 0$, $p_0 = L$ and $\det F[0] = 0$. (i_k here is not related to those in table 1.) Substituting (19) into (16) and using the elementary property of LCM, we find

$$\begin{aligned} \mathcal{P}_l &= \mathrm{LCM}\left(1, \frac{\det F}{\det F[1]}, \frac{\det F}{\det F[2]}, \ldots, \frac{\det F}{\det F[s]}\right) \\ &= \mathrm{LCM}\left(1, \frac{\det F}{\det F[1]}, \frac{\det F}{\det F[2] - \det F[1]}, \ldots, \frac{\det F}{\det F[t+1] - \det F[t]}\right) \\ &= \mathrm{LCM}\left(1, \bigcup_{k=0}^{t}{}' \frac{p_{i_{k+1}} p_{i_k}}{(i_{k+1} - i_k) p_{i_s}}\right), \end{aligned} \tag{21}$$

where $0 \leqslant t \leqslant s-1$ is the maximum integer such that $i_{t+1} > i_t$.

4. Dynamical period

In this section, we shall exclusively consider the $A_n^{(1)}$ case although parallel results are expected for $D_n^{(1)}$. When $l \to \infty$, one puts $i_k = J_k$ and $t = s - 1$ in formula (21). Eventually, the resulting expression coincides with equation (4.24) in [YYT], which gives the period of generic paths in the periodic box–ball system containing m_j solitons of length j. Here by generic is meant the absence of an 'effective translational symmetry' [YYT]. In the present framework, it corresponds to the time evolution $T_{l=\infty}$ of the basic periodic $A_1^{(1)}$ automaton, i.e., $\mathfrak{g}_n = A_1^{(1)}, \nu_j^{(a)} = L\delta_{j1}$.

To generalize such a connection, we invoke the combinatorial version of the Bethe ansatz explored in [KKR, KR]. Given a highest path, namely an element $p \in B$ such that $\tilde{e}_i p = 0$ for $1 \leqslant i \leqslant n$, one can bijectively attach the data $(m^{(a)}, r^{(a)})_{a=1}^n$ called *rigged configuration*. Here $m^{(a)} = \left(m_j^{(a)}\right)$ is a Young diagram involving $m_j^{(a)}$ rows of length j and $r^{(a)} = \left(r_j^{(a)}\right)$ stands for an array of partitions attached to each 'cliff' of $m^{(a)}$. $|m^{(a)}|$ is equal to the number of letters $a+1, a+2, \ldots, n+1$ contained in the corresponding path p. The separate data $(m^{(1)}, \ldots, m^{(n)})$ and $(r^{(1)}, \ldots, r^{(n)})$ are called *configuration* and *rigging*, respectively. They obey a special selection rule originating in the string hypothesis. Namely, $p_j^{(a)}$ defined by (11) must be non-negative and the maximum part of the partition $r_j^{(a)}$ is not greater than $p_j^{(a)}$ for any $(a, j) \in H$. It is known that $\det F > 0$ ([KN], lemma 3.7) for any configuration. Time evolutions of a highest path are not highest in general. Let $P_h(m) \subseteq B$ be the set of highest paths whose configuration is $m = (m^{(1)}, \ldots, m^{(n)})$.

Conjecture 1. *For a highest path $p \in P_h(m)$, suppose that $T_l^k(p)$ exists for any $k \in \mathbb{Z}_{\geqslant 1}$. Then the dynamical period of p (minimum positive integer k such that $T_l^k(p) = p$) is equal to $\mathcal{P}_l$ (16) generically, and its divisor otherwise.*

Naturally, we expect $\Lambda_l^{\mathcal{P}_l} = 1$, which can indeed be verified for $A_1^{(1)}$. Conjecture 1 implies that the generic period is a function of the configuration only and does not depend on the rigging. We have abruptly combined the Bethe ansatz results in two different regimes. The first one in section 3 is at $q = 0$ [KN], whereas the second one explained here is relevant to $q = 1$ [KKR, KR]. Conjecture 1 has been confirmed for all the highest paths in $B_1^{\otimes L}$ with $L \leqslant 9$ and all the $sl_{n\leqslant 4}$ highest paths in $B_{l_1} \otimes \cdots \otimes B_{l_L}$ with $l_1 + \cdots + l_L \leqslant 7$. It was observed that non-generic cases are pretty few and $T_l^k(p) (k \geqslant 1)$ exists for any highest p in case $B = B_1^{\otimes L}$.

Let us present a few examples of conjecture 1. To save the space, $12 \otimes 224$ is written as $12 \cdot 224$, etc, and furthermore, $\cdot$ is totally dropped for the basic periodic automata. In each table, the period under T_l with maximum l is equal to that under T_∞. The last table is a preliminary report on the $D_n^{(1)}$ case.

$A_1^{(1)}$ path = 1211122122111221122, configuration = ((32211))

l	LCM of				= period
1	1,	19,	19,	19	19
2	1,	57,	$\frac{171}{22}$,	$\frac{171}{22}$	171
3	1,	171,	$\frac{513}{22}$,	$\frac{513}{193}$	513

$A_1^{(1)}$ path $= 11 \cdot 1112 \cdot 2 \cdot 112 \cdot 122 \cdot 2 \cdot 2 \cdot 1$, configuration $= ((43))$

l	LCM of			= period
1	1,	$\frac{27}{2}$,	18	54
2	1,	$\frac{27}{4}$,	9	27
3	1,	$\frac{9}{2}$,	6	18
4	1,	9,	3	9

$A_2^{(1)}$ path $= 121121213322111133211$, configuration $= ((43111), (4))$

l	LCM of					= period
1	1,	21,	21,	21,	21	21
2	1,	$\frac{822}{29}$,	$\frac{822}{95}$,	$\frac{411}{46}$,	$\frac{411}{37}$	822
3	1,	$\frac{959}{22}$,	$\frac{959}{176}$,	$\frac{959}{169}$,	$\frac{959}{127}$	959
4	1,	$\frac{2877}{50}$,	$\frac{2877}{400}$,	$\frac{2877}{820}$,	$\frac{2877}{463}$	2877

$A_3^{(1)}$ path $= 1 \cdot 12 \cdot 3 \cdot 114 \cdot 1 \cdot 2 \cdot 22$, configuration $= ((3111), (11), (1))$

l	LCM of					= period
1	1,	$\frac{29}{5}$,	29,	$\frac{29}{4}$,	$\frac{29}{4}$	29
2	1,	$\frac{58}{7}$,	$\frac{58}{13}$,	$\frac{116}{17}$,	$\frac{116}{17}$	116
3	1,	$\frac{29}{2}$,	$\frac{29}{12}$,	$\frac{58}{9}$,	$\frac{58}{9}$	58

$D_4^{(1)}$ path $= 1 \cdot 12 \cdot 1 \cdot 223 \cdot 4\bar{2} \cdot 23 \cdot 1$, configuration $= ((431), (32), (2), (1))$

l	LCM of								= period
1	1,	$\frac{39}{7}$,	$\frac{234}{17}$,	$\frac{234}{17}$,	$\frac{26}{3}$,	$\frac{234}{17}$,	$\frac{117}{11}$,	$\frac{117}{11}$	234
2	1,	$\frac{39}{5}$,	$\frac{117}{20}$,	$\frac{117}{20}$,	$\frac{13}{2}$,	$\frac{117}{20}$,	$\frac{117}{19}$,	$\frac{117}{19}$	117
3	1,	13,	$\frac{26}{7}$,	$\frac{26}{7}$,	$\frac{26}{5}$,	$\frac{26}{7}$,	$\frac{13}{3}$,	$\frac{13}{3}$	26
4	1,	$\frac{39}{2}$,	$\frac{468}{71}$,	$\frac{468}{227}$,	$\frac{52}{11}$,	$\frac{468}{149}$,	$\frac{117}{31}$,	$\frac{117}{31}$	468

For instance, in the third example, configuration $= ((43111), (4))$ means that $m_1^{(1)} = 3, m_3^{(1)} = m_4^{(1)} = 1, m_4^{(2)} = 1$ and all the other $m_j^{(a)}$ s are 0.

5. Size of orbit

In this section, we only consider the basic periodic $A_n^{(1)}$ automaton, i.e., $B = B_1^{\otimes L}$. In addition to the period under the time evolutions, the Bethe ansatz at $q = 0$ also leads to a formula for the size of certain orbits in the periodic automaton. Recall the quantity

$$\Omega_L(m) = \det F \prod_{(a,j)\in H} \frac{1}{m_j^{(a)}} \binom{p_j^{(a)} + m_j^{(a)} - 1}{m_j^{(a)} - 1} \in \mathbb{Z} \tag{22}$$

obtained in [KN] (equation (3.2)) (denoted by $R(\nu, N)$ therein) as the number of off-diagonal solutions to the string centre equation. Here $\binom{s}{t} = s(s-1)\cdots(s-t+1)/t!$, and L and

$m = \big(m_j^{(a)}\big)$ enter the right-hand side through (11) and (17) with $\nu_j^{(a)} = L\delta_{a1}\delta_{j1}$. In this special case, the general identity known as the combinatorial completeness of the Bethe ansatz at $q = 0$ ([KN], corollary 5.6) reads

$$(x_1 + \cdots + x_{n+1})^L = \sum_m \Omega_L(m) x_1^{L-q_1} x_2^{q_1-q_2} \cdots x_n^{q_{n-1}-q_n} x_{n+1}^{q_n}, \qquad \left(q_a = \sum_{j\geqslant 1} j m_j^{(a)} \right). \tag{23}$$

The left-hand side is the character of $B = B_1^{\otimes L}$. The sum extending over all $m_j^{(a)} \in \mathbb{Z}_{\geqslant 0}$ cancels out except leaving the non-zero contributions exactly when $L \geqslant q_1 \geqslant \cdots \geqslant q_n$. For example, when $n = 2$, one has $\Omega_6(((3), (1))) = 6$, $\Omega_6(((21), (1))) = 36$, $\Omega_6(((111), (1))) = 18$ summing up to $\binom{6}{3,2,1} = 60$ for $(q_1, q_2) = (3, 1)$, whereas $\Omega_6(((1), (3))) = 6$, $\Omega_6(((1), (21))) = -18$, $\Omega_6(((1), (111))) = 12$ cancelling out for $(q_1, q_2) = (1, 3)$. In this sense, $\Omega_L(m)$ gives a decomposition of the multinomial coefficients according to the string pattern m. It is known ([KN], lemma 3.7) that $\Omega_L(m) \in \mathbb{Z}_{\geqslant 1}$ for any configuration, namely under the condition $p_j^{(a)} \geqslant 0$ for all $(a, j) \in H$. Moreover, it was pointed out in [KOTY] that expression (22) for $A_1^{(1)}$ simplified by (19) coincides exactly with equation (2.3) in [YYT], which is the number of automaton states that contain $m_j^{(1)}$ solitons with length j. Thus, it is natural to ask what is being counted by (22) for the basic periodic $A_n^{(1)}$ automaton in general.

To deal with this problem, we need to consider a more general class of time evolutions. Let $B^{a,j}$ be the crystal of the Kirillov–Reshetikhin module $W_j^{(a)}$ [KMN]. The crystal so far written as B_l is $B^{1,l}$ in this notation. Given a path $p \in B = B_1^{\otimes L}$, seek an element $v^{a,j} \in B^{a,j}$ such that $v^{a,j} \otimes p \simeq p' \otimes v^{a,j}$ for some p' under the isomorphism $B^{a,j} \otimes B \simeq B \otimes B^{a,j}$. If such a $v^{a,j}$ exists and p' is unique even when $v^{a,j}$ is not unique, we denote $p' \in B$ by $T_j^{(a)}(p)$. Otherwise we say that $T_j^{(a)}(p)$ does not exist. We call p *evolvable* if $T_j^{(a)}(p)$ exists for all members of $\mathcal{T} := \big\{T_j^{(a)} \mid 1 \leqslant a \leqslant n, j \in \mathbb{Z}_{\geqslant 1}\big\}$. For such a p, write $\mathcal{T}p = \bigcup_{a,j} T_j^{(a)}(p)$. We say that p is *cyclic* if all the paths $p, \mathcal{T}p, \mathcal{T}^2 p, \ldots$ are evolvable. These paths form an orbit $\mathrm{Orb}(p) := \bigcup_{t\geqslant 0} \mathcal{T}^t(p)$, which is necessarily a finite subset in B. As in the previous section, we let $P_h(m) \subseteq B = B_1^{\otimes L}$ denote the set of highest paths whose configuration is $m = (m^{(1)}, \ldots m^{(n)})$. (Thus, $P_h(m)$ is dependent on L.)

Conjecture 2. *Given a configuration $m = (m^{(1)}, \ldots, m^{(n)})$, one has two alternatives: all the paths in $P_h(m)$ are cyclic, or all the paths are not cyclic. In the former case, the following formula is valid:*

$$\Omega_L(m) = \left| \bigcup_{p\in P_h(m)} \mathrm{Orb}(p) \right|. \tag{24}$$

All the highest paths with length $L \leqslant 5$ are cyclic. The smallest example of non-cyclic $P_h(m)$ emerges at $L = 6$, which is $P_h(((22), (2)))$ only. It consists of the unique highest path $p = 112233$, which is evolvable but not cyclic. In fact, one has $T_1^{(2)}(p) = 213213$ but in the next step $[13] \otimes (213213) \simeq (311223) \otimes [13]$ whereas $[23] \otimes (213213) \simeq (223311) \otimes [23]$. Here $[13] \in B^{2,1}$ stands for the column tableau of depth 2, etc. Thus, p' in the above sense is not unique, meaning that $213213 \in \mathcal{T}p$ is not evolvable, hence p is not cyclic. For $L = 7$, again $P_h(((22), (2)))$ is the unique case consisting of non-cyclic paths. We have checked the conjecture up to $L = 8$, where there are five non-cyclic ones out of the 56 possible configurations. Some examples of conjecture 2 are presented in the following table. (We write $\mathrm{Orb}(m) = \bigcup_{p\in P_h(m)} \mathrm{Orb}(p)$, which also depends on L.)

L	m	$\Omega_L(m) = \|\mathrm{Orb}(m)\|$
6	((3))	6
6	((21), (1))	36
6	((1111), (11), (1))	12
7	((31), (1))	56
7	((221), (21), (1))	63
7	((2111), (21), (1))	133
7	((2111), (111), (11), (1))	112
8	((111111), (1111), (11))	4
8	((2211), (211), (11), (1))	192
8	((21111), (211), (11), (1))	304

For example, in the third case $L = 6, m = ((1111), (11), (1))$, one has

$$
\begin{aligned}
P_h(m) &= \{121234, 123124, 123412\},\\
\mathrm{Orb}(m) &= \{121234, 123124, 123412, 124123, 212341, 231241,\\
&\qquad 234121, 241231, 312412, 341212, 412123, 412312\}.
\end{aligned}
\tag{25}
$$

6. Summary

In this paper, we have constructed new periodic soliton cellular automata and studied them by a novel application of the Bethe ansatz. Section 2 contains the definition of the periodic automata associated with any non-exceptional affine Lie algebra $\mathfrak{g}_n$. Local states range over the crystal B_l of $\mathfrak{g}_n$ and the time evolution (1) is a translation in the extended affine Weyl group. In section 3, we have shown that Bethe eigenvalues at $q = 0$ become a $2\mathcal{P}_l$th root of unity, where $\mathcal{P}_l$ is explicitly given by formula (16). In section 4, $\mathcal{P}_l$ is conjectured to yield the dynamical period of the $A^{(1)}_n$ automata if F and $F[b, k]$ in (16) are specified by the combinatorial Bethe ansatz [KKR, KR]. In section 5, the Bethe ansatz character formula (23) is found to measure the size of orbits of the automata as in conjecture 2.

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